Valuation of Boundary-Linked Assets by Stochastic Boundary Value Problems Solved with a Wavelet-Collocation Algorithm

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Abstract—This article studies the valuation of boundary-linked assets and their derivatives in continuous-time markets. Valuing boundary-linked assets requires the solution of a stochastic differential equation with boundary conditions, which, often, is not Markovian. We propose a wavelet-collocation algorithm for solving a Milstein approximation to the stochastic boundary problem. Its convergence properties are studied. Furthermore, we value boundary-linked derivatives using Malliavin calculus and Monte Carlo methods. We apply these ideas to value European call options of boundary-linked assets.

Keywords—Stochastic boundary value problems, Financial derivatives, Wavelets, Collocation methods.

1. INTRODUCTION

In this paper we consider pricing boundary-linked assets and their derivatives in continuous-time markets. The values of these assets are contractually linked at several dates by means of boundary constraints. Therefore, valuing boundary-linked assets requires the solution of boundary value stochastic differential equations.

A stochastic boundary value problem (BVP) is defined as

$$dX_t = b\,(t, X_t)\; dt + \sigma\,(t, X_t)\; dW_t, \qquad \text{for } t \in [0, T], \tag{1}$$

with a boundary condition

$$\alpha(X) = c, \tag{2}$$

where W_t is a d-dimensional Brownian motion, X_t a continuous time d-dimensional stochastic process, α a continuous linear operator from the trajectories' space to $\mathbb{R}^d$, and $c \in \mathbb{R}^d$ constant.

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For example, a boundary condition can be a terminal condition with $\alpha(X) = X_T$, $\alpha(X) = A_0X_0 + A_TX_T$ where A_1 and A_T are real matrices, or a more involved condition such as $\alpha(X) = \int_0^T dA_tX_t$, where A_t is a $d \times d$ matrix whose components are functions of bounded variation in $[0, T]$. The theory of stochastic BVPs has also considered some cases of nonlinear operators α, see, e.g., [1]. Other references on boundary value stochastic differential equations are [2,3] and [4].

Stochastic BVPs typically arise from the application of the Pontryagin's maximum principle to stochastic control optimum problems with finite time horizon T, where the boundary condition is given by the transversality condition (see, e.g., [5, Proposition 10.1, pp. 112–113]). These systems cannot usually be analytically solved, and algorithmic tools are required to cope with these problems. Despite recent contributions in stochastic BVP literature (see, e.g., [6] and [7], that focus on Stratonovich integrals), much can be done to enlarge the catalogue of techniques for solving BVP.

In this paper, we propose a projection-based method for solving stochastic BVPs. Its main idea consists of using a wavelet-collocation method to solve a finite-difference Milstein approximation to the stochastic differential problem. We prove that this procedure provides a strong approximation for the solution to (1) and (2). We study the numerical performance of the algorithm in several examples.

We apply these ideas to study the valuation of boundary-linked assets and their derivatives. The analysis of boundary-linked assets is not only a theoretical problem, but can also be applied to the increasingly exotic assets traded in actual economies. With the growing sophistication of financial markets, investors are demanding new, more complex options products, tailored to their needs. In particular, there is an increasing number of financial assets whose values are contractually linked at certain periods of time, such as leases and rental agreements. An illustrative example is the English real estate lease market. In English law, two legal estates exist in buildings and land: freehold (absolute ownership which does not expire) and leasehold (temporal possession for a specified time period). Leasehold enables liability on covenants to pass from tenant to tenant and indeed from landlord to landlord. In this context, the lessor bears the risk associated with the residual market of the asset at the maturity date of the contract, and the buyer bears the short-term lease risk, where the value fluctuation of the lease randomly fluctuates subject to some boundary constraints, e.g., a zero value of the leasing contract at the maturity date. The value of lease assets can be formulated by a second-order boundary value stochastic differential equation

$$\ddot{X}(t) = b(t)\dot{X}(t) + \dot{W}(t), \qquad t \in [0, T], \tag{3}$$

$$X(0) = \rho, \qquad X(T) = 0; \tag{4}$$

that is, the acceleration of lease assets prices is proportional to their growth rate and affected by a white noise shock. Note that any second-order problem can be reduced to a first-order system of stochastic differential equations with boundary value conditions in the space of states, see, e.g., [4]. Hence, to value boundary-linked assets, we are faced with the problem of solving stochastic differential equations with boundary conditions.

Often, the solution of stochastic BVPs does not satisfy the Markovian conditional independence property, see, e.g., [8] and [9]. Therefore, standard Black-Scholes arguments cannot sometimes be applied to value derivatives of boundary-linked assets. We propose the use of Malliavin calculus to value these derivatives. In particular, we consider the generalized Clark-Ocone formula and present a procedure for its computation based on the Monte Carlo method and wavelets approximations. To illustrate this methodology, we consider a European call option of boundary-linked assets.

The rest of the paper is organized as follows. After some preliminaries, Section 2 provides a brief introduction to stochastic BVPs. In Section 3 we present an algorithm for solving boundary value stochastic differential problems. Its numerical performance is illustrated by means of some examples. Next we study the properties of the solution approximation. In Section 4 we apply these ideas to value boundary-linked assets, of which prices are determined by a stochastic differential equation with boundary conditions. Also, we consider the valuation of boundary-linked derivatives and their numerical computation. Finally, proofs are placed in Appendix A and a MATLAB code is included in Appendix B.

2. STOCHASTIC BVPS: PRELIMINARIES

Now we introduce some basic notation and tools that will be used through the paper.

White Noise Process

Let $\mathcal{S}$ be the Schwartz space in $\mathbb{R}$ and let $\mathcal{S}'$ be its dual (the space of tempered distributions) endowed with the weak-* topology and Borel subsets $\mathcal{B}$. By the Minlos theorem, there is a probability measure μ on $\mathcal{S}'$ such that $\int_{\mathcal{S}'} \exp\{i\langle\omega,\phi\rangle\}\, d\mu(\omega) = \exp\{-(1/2)\|\phi\|^2_{L_2(\mathbb{R})}\}$ for all $\phi \in \mathcal{S}$, where $\langle\omega,\phi\rangle$ is the evaluation of $\omega \in \mathcal{S}'$ on ϕ. The space $(\mathcal{S}',\mathcal{B},\mu)$ is the white noise probability measure, satisfying the Itô isometry $E_\mu[\langle\omega,\phi\rangle] = \|\phi\|^2_{L_2(\mathbb{R})}$ for all $\phi \in \mathcal{S}$. Consider d independent realizations from μ, then we construct a d-dimensional Wiener process $W_t = (\langle\omega_1, I_{[0,t]}\rangle, \ldots, \langle\omega_d, I_{[0,t]}\rangle)'$, which has a continuous modification in $C(\mathbb{R};\mathbb{R}^d)$ with $W_t = 0$ for $t \leq 0$, and such that $\langle\omega,\phi\rangle = \int_{\mathbb{R}} \phi(t)\, dW_t$ for all $\phi \in \mathcal{S}$, in the sense of Itô's integral. Consider the Gelfand triple $\mathcal{S} \subset L_2(\mu) \subset \mathcal{S}'$, where

$$L_2(\mu) = \left\{X : \mathcal{S} \to \mathbb{R} : \|X\|^2_{L_2(\mu)} = \int_{\mathcal{S}'} \langle X,\omega\rangle^2\, d\mu(\omega) < \infty\right\}. \tag{5}$$

An orthogonal basis for $L_2(\mu)$ is given by the family $\{H_k\}$, indexed by all vectors $k = (k_1, \ldots, k_m)$, with $\{k_j\}_{j=1}^m \subset \mathbb{N}$, for all $m = 1, 2, \ldots$, where $H_k(\omega) = \prod_{j=1}^m h_{k_j}(\langle\omega, e_j\rangle)$ and $\{h_n\}$, $\{e_n\}$ are Hermite polynomials and Hermite functions, respectively. Then, we define a singular white noise generalized process as follows:

$$\dot{W}_t = \left(\dot{W}_t(\omega_1), \ldots, \dot{W}_t(\omega_d)\right)', \tag{6}$$

with $\dot{W}_t(\omega) = \sum_k e_k(t) H_k(\omega)$. A detailed review of this topic can be found, e.g., in [10] and [11].

Let $\Omega = C_0([0,T];\mathbb{R}^d)$ be the space of all the continuous functions in $[0,T]$ which vanish at zero, with $T > 0$ deterministic. The restriction of the Wiener process W_t to $[0,T]$ induces a Borel probability space, whose completion is denoted by $(\Omega, \mathcal{A}, P)$. Let $\{\mathcal{A}_t\}$ denote the filtration generated by W_t, completed and made right continuous.

Malliavin Calculus

Next we introduce some tools from Malliavin calculus. For all $h \in L_2([0,T];\mathbb{R}^d)$, consider $W(h) = \int_0^T h_s\, dW_s$. Let $C^\infty(\mathbb{R}^n)$ be the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ infinitely continuously differentiable such that f and all its derivatives are bounded. We denote by $\mathcal{D}$ the set of real random variables of the form $F = f(W(h_1), \ldots, W(h_h))$, with $f \in C^\infty(\mathbb{R}^n)$ for any $n \in \mathbb{N}$ and $h_1, \ldots, h_n \in L_2([0,T];\mathbb{R}^d)$. For all $F \in \mathcal{D}$ we can define the differential operator DF as the stochastic process

$$D_t F = \sum_{j=1}^n \frac{\partial}{\partial x_j} f\left(W(h_1), \ldots, W(h_h)\right) h_j(t), \qquad \forall\, t \in [0,T], \tag{7}$$

and the iterated differential $D^n_{t_1,\dots,t_n}F = D_{t_1}\cdots D_{t_n}F$, and $D^0F = F$. Let $\mathbb{D}^{q,p}$ be the closure of $\mathcal{D}$ with respect to the Sobolev norm

$$\begin{aligned}\|F\|_{q,p} &= \left(\|F\|^p_{L_p(\Omega)} + \sum_{j=1}^{n}\left\|\left\|D^j_{t_1,\dots,t_j}F\right\|_{L_2[[0,T]^j]}\right\|^p_{L_p(\Omega)}\right)^{1/p}\\ &= \left(E\left[|F|^p\right] + \sum_{j=1}^{n}E\left[\left(\int_{[0,T]^j}\left|D^j_{t_1,\dots,t_j}F\right|^2 dt_1\cdots dt_j\right)^{p/2}\right]\right)^{1/p},\end{aligned} \tag{8}$$

with $p \in (0,\infty)$, and $\mathbb{D}^{q,\infty}$ as the elements F in $\mathbb{D}^{q,2}$ with finite norm

$$\|F\|_{q,\infty} = \|F\|_{L_\infty(\Omega)} + \sum_{j=1}^{n}\left\|\left\|D^j_{t_1,\dots,t_j}F\right\|_{L_2[[0,T]^j]}\right\|_{L_\infty(\Omega)}. \tag{9}$$

We also define $\mathbb{D}^{\infty,p} = \bigcap_{q\geq 1}\mathbb{D}^{q,p}$ and $\mathbb{D}^{\infty} = \bigcap_{q,p\geq 1}\mathbb{D}^{q,p}$.

The operator $D : \mathbb{D}^{1,2} \subset L_2(\Omega) \to L_2([0,T]\times\Omega)$ is known as the Malliavin derivative of $F \in \mathbb{D}^{1,2}$, with first-order derivatives and second-order moments. The adjoint operator of D, denoted by δ, is defined for all processes u such that $E[\int_{[0,T]} D_tFu_t\,dt] \leq c\|F\|_{L_2(\Omega)}$. If $u \in \mathrm{Dom}(\delta)$, then $E[\delta(u)F] = E[\int_{[0,T]} D_tFu\,dt]$ for all $F \in \mathbb{D}^{1,2}$. Often, the operator δ is expressed as $\delta(u) = \int_0^T u_t\,dW_t$, and is known as the Skorohod integral. It can be proved that $\delta(u)$ is equal to the Itô integral if the process u_t is adapted. The duality can be used to establish the Clark-Ocone formula, see, e.g., [12]; that is for all $F \in \mathbb{D}^{1,2}$, $F = E[F] + \int_0^T E[D_tF \mid \mathcal{A}_t]\,dW_t$. For an introduction to the Malliavin calculus and its properties, see, e.g., [13–15] and [16]. Malliavin derivatives can be also considered as Fréchet derivatives, see, e.g., [16]. Malliavin calculus can also be introduced using Wiener-Itô chaos expansions, see, e.g., [17].

Stochastic BVP Solutions

Consider the stochastic BVP (1) and (2). Although the study of existence of solutions for these problems is beyond the scope of this paper, we will sketch a proof using a scheme similar to that of the deterministic case. Notice that there exists a unique solution associated to $Dx(t) = 0$ with $\alpha(x) = c$ since α are linearly independent (at least over $\mathrm{Ker}\{D\}$). Consider a Green's matrix of functions $G(t,s)$, such that any $g \in C_0([0,T];\mathbb{R}^d)$ with Dg integrable can be expressed as follows:

$$g(t) = P_0(g)(t) + \int_0^T G(t,s)Dg(s)\,ds, \tag{10}$$

where $P_0(g)$ is the unique element in $\mathrm{Ker}\{D\}$ which agrees with $\alpha(g)$. Furthermore,

$$Dg(t) = D_tP_0(g)(t) + \int_0^T D_tG(t,s)g(s)\,ds. \tag{11}$$

Let $\mathcal{H}_{1,2} = \{h \in L_2([0,T]\times\Omega;\mathbb{R}^d) : \exists o \in (\mathbb{D}^{1,2})^d,\ Do = h\}$. We will express the stochastic BVP (1) and (2) in a more convenient way, using the following property: define $Dx = u$, thus $u = G[x]$ and $G^{-1}[u] = x$, reciprocally, with

$$G[x](t) = P_0(x)(t) + \int_0^T G(t,s)x(s)\,ds, \tag{12}$$

$$G^{-1}[u](t) = D_t\{P_0(x)(t)\} + \int_0^T D_tG(t,s)x(s)\,ds. \tag{13}$$

Therefore, defining $U = G(X)$, and the nonlinear operator

$$T[U](t) := b\left(t, G^{-1}[U](t)\right) + \sigma\left(t, G^{-1}[U](t)\right) \dot{W}_t, \tag{14}$$

we can express the stochastic BVP as $U = T[U]$. Then, we can guarantee the existence of solution in BVP by proving the existence of a fixed point U^0 for T, for which it suffices that T is a continuous retraction in a space isometric to $\mathcal{H}_{1,2}$, and a pathwise unique solution U^0 exists. Also, $X^0 = G^{-1}(U^0)$ is the almost sure (a.s.) unique solution of BVP, with $U_t^0 = G(U^0)(t)$ the Malliavin derivative of X_t^0. Since

$$\begin{aligned} E\left[\|T[U]\|^2_{L_2[0,T]}\right] &= E\left[\int_0^T \left\|b\left(t, G^{-1}[U](t)\right) + \sigma\left(t, G^{-1}[U](t)\right) \dot{W}_t\right\|^2 dt\right] \\ &= \int_0^T E\left[\left\|b\left(t, G^{-1}[U](t)\right)\right\|^2 + \left\|\sigma\left(t, G^{-1}[U](t)\right)\right\|^2\right] dt, \end{aligned} \tag{15}$$

T is contractive if $\|b(t,x)\| \leq k\|x\|$, $\|\sigma(t,x)\| \leq c\|x\|$, for all t, x, and $(k+c)\|G^{-1}\| < 1$. Note that the solution X^0 satisfies

$$X_t^0 = X_0^0 + \int_0^t b\left(s, X_s^0\right) ds + \int_0^t \sigma\left(s, X_s^0\right) dW_s, \tag{16}$$

for some initial condition X_0^0. Given X^0, we can define a sequence $\{X_t^n\}$ as follows:

$$X_t^{n+1} = X_0^0 + \int_0^t b\left(s, X_s^n\right) ds + \int_0^t \sigma\left(s, X_s^n\right) dW_s, \tag{17}$$

where $E[\int_0^T |X_t^n - X_t^0|^2\, dt] \to 0$ under appropriate Lipschitz conditions on b and σ. Hence, using (17), the existence of a unique continuous version of X^0 on $[0,T]$ can be proved using arguments similar to the case of ordinary stochastic differential equations.

Sufficient conditions for fixed points can be restrictive and if no solution exists, a weaker concept can be used by generalizing the notion of fixed point. For a $\delta > 0$, a point U_0 is called a δ-point of T if $\|T(U_0) - U_0\| < \delta$ and a set of δ-points is called a δ-set (see [18, p. 1553]). Therefore we can consider a set $X_0 = G^{-1}(U_0)$ of δ-equivalent solutions which may exist even when the exact solution does not.

Adaptativeness

The solutions of stochastic BVPs are anticipative in nature due to the boundary condition. But given an appropriate (anticipating) initial condition, the dynamics of the process is driven by an ordinary stochastic differential equation. This is the logic underlying shooting numerical methods. These methods are widely used to solve deterministic BVP (see [19] for a review), and have been recently extended by [6] to solve Stratonovich stochastic BVPs. A shooting method is a successive substitution method based on the idea of guessing the initial condition until its associate solution satisfies the boundary condition.

We use the shooting argument to define a conditional adaptativeness for solutions of BVPs. Let $\mathcal{A}_0$ be the completion of the smallest σ-algebra such that $\{\alpha(X^0)\}$ is measurable, and consider the filtration $\{\mathcal{F}_t\}_{t>0}$ with $\mathcal{F}_t = \mathcal{A}_0 \cap \mathcal{A}_t$. Note that conditioning on $\{\alpha(X^0) = c\}$, the unique solution X_t^0 satisfies (16), where X_0^0 is $\mathcal{A}_0$ measurable, and as a consequence X^0 is adapted respect to $\{\mathcal{F}_t\}_{t>0}$. Therefore, conditioning on the boundary condition expression (1) can be considered as an Itô integral or, alternatively, a generalized process (see [11]). Otherwise, equation (1) should be interpreted in terms of Skorohod stochastic integrals.

3. AN ALGORITHM TO SOLVE STOCHASTIC BVPS

The numerical resolution of stochastic BVPs is the aim of this section. We propose a wavelet projection-based algorithm for solving stochastic differential equations with boundary conditions. Its main idea consists of using a wavelet-collocation method to solve a finite-difference approximation to the stochastic BVP. With this end in view, we first introduce some concepts of wavelet approximation.

Within the last decades, wavelet multiresolution methods have proved to be a flexible method for approximating relatively irregular functions with a parsimonious number of parameters. The first wavelet basis can be at least traced to the [20] work, but the theoretical foundations of wavelets have been established by physicists and mathematicians from the early 1930s to the 1980s. The interest on wavelets has increased since [21] and [22] introduced the use of multiresolution as a framework to study wavelets expansions. A historical perspective can be found in [23] and [24]. Excellent monographs in wavelets are [22–25] and [26].

Given the Hilbert space $L_2(\mathbb{R})$, let consider a sequence of closed subspaces $\{V_n\}_{n\in\mathbb{Z}}$ such that:

(i) $V_n \subset V_{n+1}$, $\forall n \in \mathbb{Z}$,
(ii) $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$, and
(iii) $\bigcup_{n\in\mathbb{Z}} V_n$ is dense in $L_2(\mathbb{R})$.

In particular we say that $\{V_n\}_{n\in\mathbb{Z}}$ is a multiresolution if each subspace V_n is the span of an orthonormal basis $\{\phi_{n,k}\}_{k\in\mathbb{Z}}$, with $\phi_{n,k}(t) = 2^{n/2}\phi(2^n t - k)$, and $\phi \in L_2(\mathbb{R})$ is known as the father wavelet. This concept was introduced by [21].

As $\{\phi_{n,k}\}_{k\in\mathbb{Z}}$ are orthonormal, if $\Pi_{V_n}(x)$ denotes the orthogonal projection of an arbitrary $x \in L_2(\mathbb{R})$ into V_n, then

$$x(t) = \lim_{n\to\infty} \Pi_{V_n}(x)(t) = \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \langle x, \phi_{n,k}\rangle_{L_2} \phi_{n,k}(t), \tag{18}$$

in the sense of L_2. Whenever ϕ has compact support, for each $t \in \mathbb{R}$ the summation in (18) contains a finite number of nonnull terms. Otherwise it should be truncated for practical applications. In practice, one of the most popular wavelets systems is the compactly supported wavelet proposed by Daubechies, see [23].

Due to the fact $V_0 \subset V_1$, any father wavelet can be expressed as

$$\phi(t) = \sum_{k\in\mathbb{Z}} a_k \phi(2t - k), \tag{19}$$

for some $\{a_k\}_{k\in\mathbb{Z}} \in l_2$. Taking Fourier transforms, we can express (19) as

$$\Phi(\omega) = A\left(\frac{\omega}{2}\right)\Phi\left(\frac{\omega}{2}\right), \tag{20}$$

where Φ is the Fourier transformed of ϕ and $A(\omega) = (1/2)\sum_{k\in\mathbb{Z}} a_k e^{-ik\omega}$. A multiresolution can be defined by finding a function $A(\omega)$, which means finding a sequence $\{a_k\}$, such that the Fourier inverse of Φ satisfying (20) is a father wavelet. Daubechies proposed a procedure to construct a finite sequence $\{a_k\}$ so that

$$\phi(t) = \sum_{k=1}^{2N-1} a_k \phi(2t - k) \tag{21}$$

and the resulting ϕ has compact support and $2N - 1$ vanishing moments (in other words, N determines the number of nonzero coefficients in (18) and is an index of the Daubechies wavelets). Furthermore, the coefficients $\{a_k\}_{k=1}^{2N-1}$ can be computed recursively. For a detailed exposition about Daubechies wavelets of order N and their properties, see [23].

Let $W_2^r(\mathbb{R})$ be the Sobolev space of functions (a.s. identical) with L_2-integrable weak derivatives up to order r. If $x \in W_2^r(\mathbb{R})$, under appropriate conditions, wavelets derivatives can also approximate the weak derivatives of x. The multiresolution ideas can be specialized to the space $L_2([0,T])$ taking a multiresolution $\{V_n\}_{n\geq 0}$. In this context, it can be proved that $\Pi_{V_n}(x) \to x$ uniformly for all $x \in C([0,T])$, see, e.g., [27].

The first basic step of our algorithm is to consider a real wavelet multiresolution $\{V_n\}_{n=1}^{\infty}$ in $L_2([0,T])$. To simplify notation throughout the remainder of the paper, given a vector of d functions $x(t) = (x_1(t), \ldots, x_d(t))'$, we will denote the wavelet approximation of any $x \in L_2(\mathbb{R})^d$ by

$$\Pi_{V_n}(x)(t) = \sum_{k\in\mathbb{Z}} \theta_{n,k}\phi_{n,k}(t), \tag{22}$$

where $\theta_{n,k} \in \mathbb{R}^d$ is a vector of coefficients. Henceforth we will consider compactly supported wavelets such as the ones proposed by Daubechies; otherwise we should truncate the summation in (22).

The next step is to consider a finite-difference approximation to the stochastic differential equation. For the sake of simplicity, we first consider the problem of solving an autonomous stochastic system $dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t$, with $\alpha(X) = c$. In particular, we consider Milstein's [28] finite-difference approach,

$$\begin{aligned} X_n(t_{i,n}) - X_n(t_{i-1,n}) &= h_n b\left(X_n(t_{i-1,n})\right) + \sigma\left(X_n(t_{i-1,n})\right)\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right) \\ &+\sigma\left(X_n(t_{i-1,n})\right)\frac{\partial\sigma}{\partial x}\left(X_n(t_{i-1,n})\right)\left[\int_{t_{i-1,n}}^{t_{i,n}}\int_{t_{i-1,n}}^{s_1} dW_{s_1}\,dW_{s_2}\right], \end{aligned} \tag{23}$$

$$X_n(0) = c. \tag{24}$$

The double stochastic integral can be readily computed, e.g., in the scalar case

$$\int_{t_{i-1,n}}^{t_{i,n}}\int_{t_{i-1,n}}^{s_1} dW_{s_1}\,dW_{s_2} = \frac{1}{2}\left(\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right)^2 - h_n\right). \tag{25}$$

For the multivariate case see [29, Chapter 5, Section 8].

Thus, the third and final step of the algorithm consists of applying the wavelet-collocation method to the Milstein approximation and solving the following system of nonlinear equations in $\theta_{n,k} \in \mathbb{R}$:

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \theta_{n,k}\left(\phi_{n,k}(t_{i,n}) - \phi_{n,k}(t_{i-1,n})\right) &= h_n b\left(X_n(t_{i-1,n})\right) \\ &+\sigma\left(\sum_{k\in\mathbb{Z}}\theta_{n,k}\phi_{n,k}(t_{i-1,n})\right)\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right) \\ +\frac{1}{2}\sigma\left(\sum_{k\in\mathbb{Z}}\theta_{n,k}\phi_{n,k}(t_{i-1,n})\right)\frac{\partial\sigma}{\partial x}&\left(\sum_{k\in\mathbb{Z}}\theta_{n,k}\phi_{n,k}(t_{i-1,n})\right)\left(\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right)^2 - h_n\right), \end{aligned} \tag{26}$$

$$\sum_{k\in\mathbb{Z}} \theta_{n,k}\alpha\left(\phi_{n,k}\right) = c, \tag{27}$$

where $t_{i,n} = 2^{-n}i \in [0,T]$ with $i \in \mathbb{Z}$, and $h_n = 2^{-n}$. The solution coefficients $\{\theta^*_{k,n}\}$ determine $X_n \in V_n$ as

$$X_{\theta^*,n}(t) = \sum_{k\in\mathbb{Z}} \theta^*_{n,k}\phi_{n,k}(t), \tag{28}$$

where only a finite number of terms are different from zero.

Often system of equations (26) and (27) is nonlinear and has to be solved by numerical methods. There are numerous methods for solving nonlinear equations (see, e.g., [30]). However, we consider Newton's method for systems of nonlinear equations (see [31, pp. 86–92]) because of its rapid rate of convergence for solving large systems of smooth nonlinear equations. Denote by $F(\theta) = 0$ the system of equations (26) and (27). In essence, the Newton algorithm consists of computing a search direction $\Delta\theta$ so that $\nabla F(\theta_l)\Delta\theta = -F(\theta_l)$, and updating $\theta_{l+1} = \theta_l + \Delta\theta$ iteratively from a starting guess θ_0.

Unfortunately Newton's method only ensures local convergence. In the (possible) presence of multiple local optima, an obvious probabilistic global search procedure is to use a local algorithm starting from several points distributed over the whole optimization region. But this procedure is extremely inefficient as when many starting points are used, the same minimum will eventually be determined several times. In the last decades the development of mathematical programming theory has led to significant advances in global optimization related modelling, algorithms, and real-world applications. In particular, global optimization procedures have been considered in the literature which could be used for solving system of equations (26) and (27) such as the delta algorithm [32, Chapter 10]. Alternatively, an implementation of the cubic algorithm in MAPLE to compute global optima can be found in [33].

Equation systems derived from collocation methods are often ill-posed (although in Examples I and II, the condition number of the Jacobian matrix ∇F at the numerical solution is moderate). In the case of ill-conditioned problem Tikhonov's [34] regularization method can be considered, which is equivalent to consider the Levenberg-Marquardt method (see [31, p. 227]) instead of the Newton algorithm.

Solving BVPs with $\sigma(t) = \sigma$ for all t is particularly easy. In this case, Milstein's equations are reduced to the Euler-Maruyama approximation (see [35]) and the system of equations (26) and (27) in $\theta_{n,k} \in \mathbb{R}$ to be solved is

$$X_n\left(t_{i,n}\right) - X_n\left(t_{i-1,n}\right) = \left(t_{i,n} - t_{i-1,n}\right) b\left(X_n\left(t_{i-1,n}\right)\right) + \left(W_{t_{i,n}} - W_{t_{i-1,n}}\right), \tag{29}$$

$$\alpha\left(X_n\right) = c, \tag{30}$$

where $X_n(t_{i,n}) = \sum_{k\in\mathbb{Z}} \theta_{n,k}\phi_{n,k}(t_{i,n})$, $t_{i,n} = 2^{-n}i \in [0, T]$ with $i \in \mathbb{Z}$, and $h_n = 2^{-n}$.

Also, this method can be applied to the nonautonomous stochastic systems, $dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$, with $\alpha(X) = c$. However, instead of the Milstein equation, we should consider an expansion for nonhomogeneous stochastic differential equations, see, e.g., [29, Chapter 5, Section 5].

In case of using δ-point notion, we have a δ-set of coefficients $\{\theta^*_{k,n}\}$ which can be computed by the spherical algorithm proposed by [18].

In order to illustrate the accuracy of the method, we compute several examples of stochastic BVPs with analytical solution and use the compactly supported wavelets of Daubechies. The algorithm has been implemented and the tests have been carried out on MATLAB 6.5 on an Intel Centrino Pentium M 1.6 GHz with machine precision 10^{-16}. First we consider a very simple stochastic BVP to show how to set up parameters to compute its approximate solution. In Appendix B we present the MATLAB code to clarify exposition of the proposed method.

EXAMPLE I. Consider the problem

$$dX_t = dW_t, \qquad t \in [0, 1], \tag{31}$$

$$X_{1/2} + X_1 = 0. \tag{32}$$

This problem has a solution of the form $X(t) = -(1/2)(W_{1/2} + W_1) + W_t$.

We compute the numerical approximation of its solution for a sample path of $\{W_t\}$ using Daubechies wavelets with $N = 3$ and the step length $h = 2^{-2}$ (i.e., $n = 2$ and the number of dyadic points used is 9). To compute the solution path of this problem, we consider the Milstein approximation and we solve the system of equations (26) and (27) in $\theta_{n,k} \in \mathbb{R}$ (in this case, it is a nonhomogeneous linear system). Direct methods (usually variants of Gaussian elimination) are implemented in the core of MATLAB and are made as efficient as possible for general classes of matrices.

In order to illustrate the accuracy of the numerical solution, we compute the solution path for 400 realizations of the Brownian motion $\{W_t\}$. The average approximation error $\|X(t_{i,n}) - X_{\theta^*,n}(t_{i,n})\|_\infty = \max_{\{t_{i,n}\}} |X(t_{i,n}) - X_{\theta^*,n}(t_{i,n})|$ is 2.7717×10^{-15} and its standard deviation 1.5896×10^{-15}.

In case of being interested in a higher accuracy, we can consider a larger number n. For $n = 6$, the condition number of the Jacobian matrix ∇F at the numerical solution is 93.4161, which means that we can expect to lose approximately two digits of precision during the computation of $\{\theta^*_{k,n}\}$. Figure 1 shows the values of the actual solution and its approximation over the dyadic points $t_i = 2^{-6}i \in [0, 2]$ with $i \in \mathbb{Z}$.

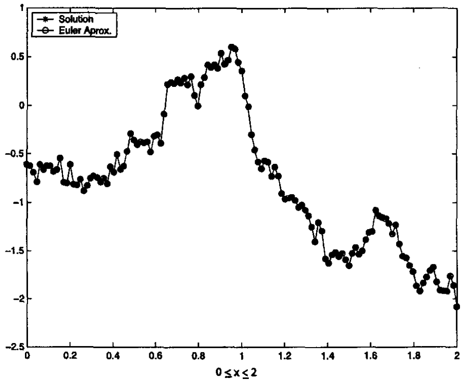


Figure 1. Numerical and exact solution of Example I with $N = 3$ and $n = 6$.

The next example is intended to demonstrate that the algorithm also behaves well in more complicated problems. However, a higher number of dyadic points (in other words, higher parameter n) should be considered to get accuracy.

EXAMPLE II. Consider the problem

$$dX_t = dW_t, \qquad t \in [0, 1], \tag{33}$$

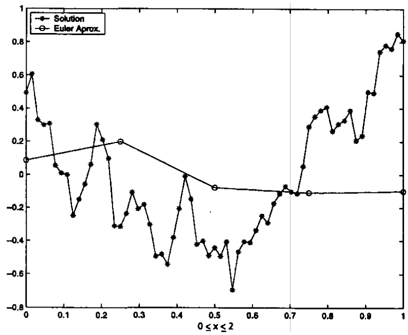
$$\int_0^1 X_t = 0. \tag{34}$$

The solution of this problem has the general form $X(t) = -\int_0^1 W_t \, dt + W_t$.

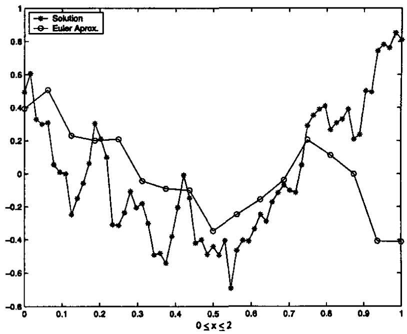
For a given sample path of $\{W_t\}$, using Daubechies wavelets with $N = 3$, Figure 2 shows the exact and the computed approximation for $n = 2, 4, 6$.

Table 1 reports the approximation errors to the solution and the computational cost for solving the stochastic BVP for $n = 2, 4, 6$. In this case, the approximation solution with $n = 6$ gives the smallest residual $\|X(t_{i,n}) - X_{\theta^*,n}(t_{i,n})\|_\infty$ as illustrated in Figure 2, for which the condition number of ∇F at the numerical solution is 620.8568, which means that we can expect to lose approximately three digits of precision during the computational process.

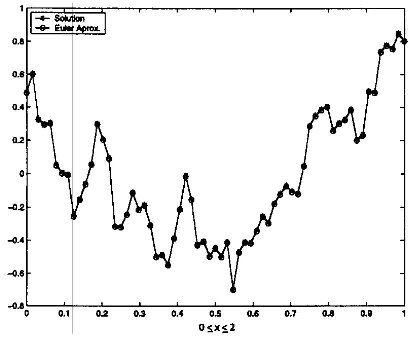
Similarly to stochastic BVPs, most stochastic differential equations arising in real-world applications cannot be solved exactly. Numerical methods to get accurate solutions are Euler-Maruyama and Milstein schemes, among others. A review of the literature can be found, e.g., [29]. The proposed algorithm can also be used to solve ordinary stochastic differential equations as the following example illustrates.



For $n = 2$.



For $n = 4$.



For $n = 6$.

Figure 2. For $n = 2, 4, 6$, numerical and exact solutions of Example II with $N = 3$.

Table 1. Approximation errors and running times for computing Example 2 with $n = 2, 5, 6$.

	$\left\| X(t_{i,n}) - X_{\theta^*,n}(t_{i,n}) \right\|_\infty$	CPU (Seconds)
$n = 2$	0.2058	0.02
$n = 4$	0.0997	0.03
$n = 6$	0.0075	4.66

EXAMPLE III. Consider the problem

$$dX_t = bX_t\,dt + \sigma X_t\,dW_t, \qquad t \in [0,1], \tag{35}$$

with $X_0 = \xi$, and $W_0 = 0$ a.e. The solution of this problem has the general form

$$X(t) = \xi \exp\left(\left(b - \frac{\sigma^2}{2}\right) t + \sigma W_t\right). \tag{36}$$

Assume that $b = 2$, $\sigma = 1$, and $\xi = 1$. We compute the numerical solution of this problem using Daubechies wavelets with $N = 3$. Figure 3 shows that the approximation error is satisfactory for $n = 6$, although there is room for improvement in the right-hand side of the time interval and the ill-posedness of ∇F (as its condition number is 30559).

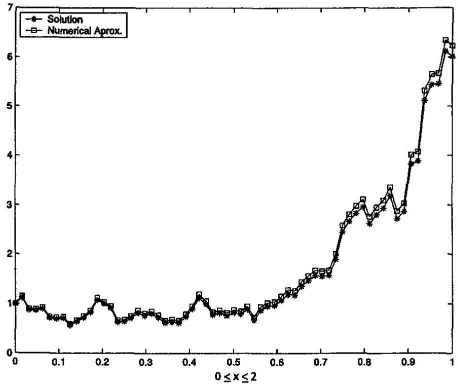


Figure 3. Numerical and exact solutions of Example III with $N = 3$ and $n = 6$.

Although we have focused on solving stochastic BVPs with linear boundary conditions, we can also apply this method to problems in which the boundary conditions are given by nonlinear continuous operators. When α is a nonlinear continuous operator, the proposed method can be applied by replacing equation (27) by $\alpha(\sum_{k\in\mathbb{Z}} \theta_{n,k}\phi_{n,k}) = c$. However, the convergence theory for this type of problem is beyond the scope of this paper.

3.1. Convergence Analysis

In this section, we study the convergence properties of the proposed method. Proofs are placed in Appendix A. Assume:

A.1. Let $\{V_n\}$ be a multiresolution in $L_2(\mathbb{R})$, with compactly supported father wavelet ϕ, and assume for all $x \in W_2^r(\mathbb{R})$, with $1 \le r \le q$, $q \ge 1$, and all integer ν, $0 \le \|\nu\|_1 \le r-1$, it is satisfied

$$\|D^\nu x - D^\nu \Pi_{V_n}(x)\|_{L_2} = O\left(2^{-(r-\|\nu\|_1)n}\right). \tag{37}$$

Whenever $x \in C^r(\mathbb{R})$ with compact support the same rates are satisfied replacing the L_2 norm by the supremum norm. In spaces $L_2([a,b])$, an analogous behavior is assumed.

There are several sufficient conditions for this result that can be found in the literature, often based on the regularity of order q assumption. The father wavelet ϕ is said to be *regular of order* $q \in \mathbb{N}$, if ϕ has a version q times continuously differentiable and for $0 \le \|\nu\|_1 \le q$, and any positive integer $p \in \mathbb{N}$, there exists a constant $C_p > 0$ such that $|D^\nu\phi(t)| < (1 + \|t\|)^{-p}C_p$, $\forall\, t \in \mathbb{R}$. See [22] for further details.

A.2. Let $X^0(t)$ be a solution of the stochastic BVP and define $\mathcal{C} = \{(t, X^0(t)')' : t \in [0,T]\}$. Also, assume that $b, \sigma \in C^2(\mathcal{N})$ where $\mathcal{N} \subset \mathbb{R}^{R+1}$ is an ε-neighborhood of $\mathcal{C}$ in the L_∞ norm, and for some $\eta > 0$, it is satisfied

$$\Pr\left\{\inf_{t\in[0,T]} \left|\det\left(I - D_x b\left(t, X^0(t)\right) - D_x\sigma\left(t, X^0(t)\right)\dot{W}_t\right)\right| > \eta\right\} = 1. \tag{38}$$

Notice that the last condition is satisfied whenever $\det(I - D_x b(t,x) - D_x\sigma(t,x)g(t))$ is nonnull for all (t,x) and for all $g \in L_1[0,T]$. For example, when $\sigma(t)$ does not depend on X it suffices $\det(I - D_x b(t,x)) \ne 0$ for all (t,x). In particular, for linear stochastic BVP, $b(t,x) = bx$ and $\sigma(t,x) = \sigma$, it suffices that $\det(I - b) \ne 0$.

We start with an auxiliary result on the rate of approximation of the wavelet-Galerkin method. Given the multiresolution $\{V_n\}$, let $x_n \in V_n$ be the wavelet-Galerkin solution to the stochastic BVP; i.e., x_n satisfies

$$\Pi_{V_n}\left\{Dx_n - b\left(t, x_n\right) - \sigma\left(t, x_n\right)\dot{W}_t\right\} = 0, \qquad \alpha\left(x_n\right) = c. \tag{39}$$

THEOREM 1. *Let consider the problem BVP with solution $X^0(t)$, and a multiresolution sequence $\{V_n\}$ in $L_2([0,T])$. Assume that A.1, A.2 are satisfied, then there exist $\delta > 0$ and an integer M such that X^0 is unique a.s. in $B(X^0, \delta) = \{X : \|X - X^0\|_\infty \le \delta\}$, and the projected system*

$$\Pi_{V_n}\left\{DX_n - b\left(t, X_n\right) - \sigma\left(t, X_n\right)\dot{W}_t\right\} = 0 \tag{40}$$

has an a.s. unique solution $X_n \in V_n \cap B(X^0, \delta)$. Furthermore, with probability one,

$$\max\left\{\left\|X_n - X^0\right\|_\infty, \left\|DX_n - DX^0\right\|_\infty\right\} = O\left(2^{-n}\right). \tag{41}$$

In order to prove the convergence of the wavelet-collocation method we will use an interpolation result.

THEOREM 2. *Consider a multiresolution $\{V_n\}$ in $L_2(\mathbb{R})$ satisfying A.1. For each $x \in L_2(\mathbb{R})$ with an almost everywhere (a.e.) continuous version with compact support, we define $\Gamma_{V_n}(x)$ as any function $g_n \in V_n$ such that $g_n(t_{n,i}) = x(t_{n,i})$, for all $\{t_{n,i} = 2^{-n}i\}_{i\in\mathbb{Z}}$. Then, there exists a unique element in $\Gamma_{V_n}(x)$. Furthermore, assuming:*

1. *ϕ is regular of order $q \ge 1$, and*
2. *the Poisson summa $\sum_{k\in\mathbb{Z}} \Phi(\omega + 2\pi k) > 0$, for almost every $\omega \in [0, 2\pi]$, being $\Phi(\omega) = \int_{\mathbb{R}} \phi(t)e^{it\omega}\, dt$ the Fourier transformed of ϕ;*

for all $x \in W_2^q(\mathbb{R})$ with compact support, there exist $K > 0$ and n_0 such that $\forall\, n > n_0$,

$$\left\|\Gamma_{V_n}(x) - x\right\|_{L_2} \le K \left\|\Pi_{V_n}(x) - x\right\|_{W_2^q}. \tag{42}$$

Given the multiresolution $\{V_n\}$, let $x_n \in V_n$ denote the wavelet-collocation solution to the stochastic BVP, and therefore

$$\Gamma_{V_n}\left\{Dx_n - b\left(t, x_n\right) - \sigma\left(t, x_n\right)\dot{W}_t\right\} = 0, \qquad \alpha\left(x_n\right) = c. \tag{43}$$

The following result is a consequence of Theorems 2 and 1.

COROLLARY 3. *Under the assumptions of Theorems 1 and 2, the wavelet-collocation method satisfies the approximation property at rate $O(2^{-n})$.*

Therefore, it remains to prove the consistency of the proposed method based on the Milstein scheme

$$\begin{aligned}\tilde{X}_n\left(t_{i,n}\right) - \tilde{X}_n\left(t_{i-1,n}\right) &= h_n b\left(\tilde{X}_n\left(t_{i-1,n}\right)\right) + \sigma\left(\tilde{X}_n\left(t_{i-1,n}\right)\right)\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right) \\ &+\sigma\left(\tilde{X}_n\left(t_{i-1,n}\right)\right)\frac{\partial \sigma}{\partial x}\left(\tilde{X}_n\left(t_{i-1,n}\right)\right)\frac{1}{2}\left(\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right)^2 - h_n\right),\end{aligned} \tag{44}$$

for $\tilde{X}_n \in V_n$.

PROPOSITION 4. *Under the assumptions of Theorems 1 and 2, let $\tilde{x}_n \in V_n$ be the approximation generated by the proposed method and x_n the solution of the wavelet-collocation method. Then, it is satisfied $\|\tilde{x}_n - x_n\|_\infty = O_p(2^{-n})$.*

4. BOUNDARY-LINKED FINANCIAL MARKETS

Consider a monetary bond and d boundary-linked assets. Let us assume that the bond has a continuous positive price per share $X_0(t)$ solving the stochastic differential equation

$$dX_0(t) = r(t)X_0(t)\,dt, \qquad X_0(0) = 1, \tag{45}$$

where $r(t)$ is a progressively measurable process satisfying $\int_0^T |r(t)|\,dt < \infty$. Therefore $X_0(t) = \exp\{\int_0^t r(s)\,ds\}$ for $t \in [0,T]$. Let $X_i(t)$ denote the price per share of the i^{th} boundary-linked asset and $X(t) = (X_1(t), \ldots, X_d(t))'$. Assume that the initial values of the boundary-linked assets $X_1(0), \ldots, X_d(0)$ are positive constants almost surely. For each $t \in [0,T]$, suppose also that these prices are governed by

$$dX(t) = b\left(t, X(t)\right)\,dt + \sigma(t)\,dW(t) \tag{46}$$

and the boundary conditions $\beta(X) = \rho$, where $\beta(X)$ is a set of d linear continuous real functionals and $\rho \in \mathbb{R}^d$. A particularly relevant example is the linear boundary value stochastic differential equations defined as

$$dX(t) = b(t)X(t)\,dt + \sigma(t)\,dW(t), \tag{47}$$

with $\beta(X) = \rho$. Following the arguments given in [9], this problem possesses a unique solution in $C^1([0,T])$ if and only if $\det\{\beta(x^s)\} \neq 0$ for some $s \in [0,T]$ (equivalently for all $s \in [0,T]$), where $x^s(t)$ is the solution of the homogeneous system $dx(t) = x(t)b(t)\,dt$ with $x(s) = I_d$; i.e., problem $dx(t) = x(t)b(t)\,dt$ with $\beta(x) = 0$ has only the trivial solution. In this case, we can express

$$X(t) = J^{-1}(t)\rho + \int_0^T G(t,x)\sigma(s)\,dW(s), \tag{48}$$

with Green function

$$G(t,s) = J^{-1}(t)\left[\int_0^s J^{-1}(u)\nu(du) - 1_{[0,s]}(t)I_d\right] J(s), \tag{49}$$

$1_{[0,s]}(t)$ being the characteristic function of the set $[0,s]$.

In the linear context, we define a portfolio $(\theta_0(t), \theta_1(t), \ldots, \theta_d(t))'$ as a progressively measurable process that represents the number of units of the assets for each $t \in [0,T]$. The value of a portfolio is given by $V_\theta(t) = \theta_0(t)X_0(t) + \theta_0(t)'X(t)$. The portfolio $\theta_0(t)$, $\theta(t) = (\theta_1(t), \ldots, \theta_d(t))'$ is called self-financing (with respect to $X_0(t)$, $X(t)$) if

$$\int_0^T \left(r(s)\theta_0(s)'X_0(s) + \theta(s)'b(s) + \sum_{k=1}^d \left|\theta(s)'\sigma_k(s)\right|^2 \right) ds < \infty, \tag{50}$$

$$dV_\theta(t) = \theta_0(t)\,dX_0(t) + \theta(t)'\,dX(t). \tag{51}$$

Then, $V_\theta(t) = V_\theta(0) + \int_0^t \theta_0(s)\,dX_0(s) + \int_0^t \theta(s)'\,dX(s)$. Notice that given an appropriate $\theta(t)'$, there exists $\theta_0(t)$ such that $(\theta_0(t), \theta(t)')$ is self-financing. A self-financing portfolio $(\theta_0(t), \theta(t)')'$ is called admissible if its corresponding process $V_\theta(t)$ is a.e. lower bounded; i.e., $\exists K_\theta > 0$ such that $V_\theta(t) \geq -K_\theta$ a.e. for all $t \in [0,T]$. This is a natural constraint in real life as debts cannot infinitely increase. An admissible portfolio is called an arbitrage in the considered market, if the associated value process satisfies $V_\theta(0)$ and $V_\theta(T) \geq 0$ a.e. with $P(V_\theta(T) > 0) > 0$.

In practice, the value of a portfolio is often discounted at the nonrisk rate. This means that we can normalize prices by defining $\tilde{X}_0(t) = 1$ and $\tilde{X}_k(t) = X_0(t)^{-1}X_k(t)$. Given a self-financing portfolio, the discounted values are $\tilde{V}_\theta(t) = L_0(t)^{-1}V_\theta(t)$, and applying the Itô formula $d\tilde{V}_\theta(t) = \theta(t)'d\tilde{L}(t)$. Therefore, the self-financing portfolio property is not affected by the discount normalization. Furthermore, if $(\theta_0(t), \theta(t)')$ is admissible for $(L_0, L(t))$, then $(\theta_0(t), \theta(t)')$ is also admissible for the normalized market $(1, \tilde{L}(t))$, as $r(t)$ is bounded.

4.1. Valuation of Boundary-Linked Derivatives

In this section we consider pricing of boundary-linked derivatives. In this context, standard Black-Scholes techniques cannot help to value derivatives of boundary-linked assets as these processes are not Markovian. However, an alternative approach based on the generalized Clark-Ocone formula can be considered. We illustrate this approach considering a European call option of boundary-linked assets.

Let X_T be the values of the d boundary-linked assets at the maturity date of the contract. By the Clark-Ocone formula,

$$X_T(\omega) = E[X] + \int_0^T E\left[D_t X_T(\omega) \mid \mathcal{F}_t\right] dW_t, \tag{52}$$

with $D_t X_T(\omega)$ is the Malliavin derivative of $X_T(\omega)$. The Clark-Ocone formula can be extended to study $\mathcal{F}_T$ random variables $G(\omega)$ that are stochastic integrals respect to processes

$$\tilde{W}_t(\omega) = \int_0^T \alpha(s,\omega)\, ds + W_t(\omega), \tag{53}$$

where $\alpha(s,\omega)$ is an $\mathcal{F}_t$ adapted stochastic process satisfying some appropriate regularity conditions. By the Girsanov's theorem, $\tilde{W}_t$ is a Wiener process under certain probability measure Q on $\mathcal{F}_T$, where $dQ(\omega) = Z_T(\omega)\, dP(\omega)$, with

$$Z_t(\omega) = \exp\left\{-\int_0^t \alpha(s,\omega)\, ds - \int_0^t \alpha(s,\omega)^2\, ds\right\}. \tag{54}$$

The generalization of the Clark-Ocone formula ensures that if $G(\omega)$ is a regular stochastic integral respect to $\tilde{W}_t$, then

$$G(\omega) = E_Q[G] + \int_0^T \varphi_Q(t,\omega)\, d\tilde{W}_t, \tag{55}$$

where $\varphi_Q(t,\omega) = E_Q[D_t G - G \int_0^T D_t\alpha(s,\omega)\, d\tilde{W}_s \mid \mathcal{F}_t]$. The proof and other technical details can be found in [16].

This result can be applied to the valuation of derivatives in linear boundary-linked markets. As the value of a portfolio is given by $V_\theta(t) = \theta_0(t)X_0(t) + \theta(t)'X(t)$, we have

$$\theta_0(t) = X_0(t)^{-1}\left(V_\theta(t) - \theta(t)'X(t)\right). \tag{56}$$

If the portfolio is self-financing,

$$dV_\theta(t) = \theta_0(t)\, dX_0(t) + \theta(t)'\, dX(t), \tag{57}$$

using (45), (47), and (56), we obtain that

$$dV_\theta(t) = \left(r(t)V_\theta(t) + (b(t) - r(t))\,\theta(t)'X(t)\right)\, dt + \theta(t)'\sigma(t)\, dW(t). \tag{58}$$

Assuming that the solution of the boundary value problem is $\mathcal{F}_t$ adapted, our aim is to find a portfolio $\theta(t)$ leading to the lower bounded $\mathcal{F}_T$ measurable random variable $G(\omega)$, such that $G(\omega) = V_\theta(T)$ and the initial value is $V_\theta(0)$.

If $V_\theta(t)$ is $\mathcal{F}_t$ adapted, taking $\alpha(t) = (b(t) - r(t))\sigma(t)^{-1}$ and $\tilde{W}_t = \int_0^T \alpha(s)\, ds + W_t$, we can express

$$dV_\theta(t) = r(t)V_\theta(t)\, dt + \theta(t)'\sigma(t)\, d\tilde{W}(t). \tag{59}$$

Therefore, the discounted portfolio $\tilde{V}_\theta(t) = X_0(t)^{-1}V_\theta(t)$ satisfies

$$d\tilde{V}_\theta(t) = X_0(t)^{-1}\theta(t)'\sigma(t)\, d\tilde{W}(t). \tag{60}$$

By the generalized Clark-Ocone theorem, the discounted final value

$$G := \tilde{V}_\theta(T) = X_0(T)^{-1}V_\theta(T) \tag{61}$$

verifies

$$G = E_Q[G] + \int_0^T E_Q\left[D_tG - G\int_0^T D_t\alpha(s)\,d\tilde{W}_s \,\Big|\, \mathcal{F}_t\right] d\tilde{W}_t. \tag{62}$$

As a consequence, $\tilde{V}_\theta(0) = E_Q[G]$ and the required portfolio is

$$\theta(s) = X_0(t)\sigma(t)^{-1}E_Q\left[D_tG - G\int_0^T D_t\alpha(s)\,d\tilde{W}_s \,\Big|\, \mathcal{F}_t\right]. \tag{63}$$

This expression can be applied to the analysis of derivative prices in boundary-linked markets in an analogous way to the Black-Scholes formula.

For example, consider a European call option which gives the owner the right to buy the stock with value X_T at exercise price p. Then, $G = (X_T - p)^+$ represents the payoff at time T. Clearly, $G = f_p(X_T)$ where $f_p(x) = (x-p)^+$. Note that $f_p(\cdot)$ is continuous but not differentiable at $x = p$ and D_tG cannot be obtained applying the chain rule. However, $f_p \in C([0,T])$ can be approximated by a sequence $\{f_n\} \subset C^1([0,T])$ with $f_n(x) = f_p(x)$ for $|x-p| \geq 1/n$, and $0 \leq f_n \leq 1$. Taking $G_n = f_n(X_T)$ we have

$$D_tG = \lim_{n\to\infty} D_tG_n = I_{[p,\infty)}(X_T)\cdot D_t X_T = I_{[p,\infty)}(X_T)\cdot X_T\cdot\sigma(t). \tag{64}$$

Hence,

$$\theta(t) = X_0(t)\sigma(t)^{-1}E_Q\left[I_{[p,\infty)}\left(X_T\right)X_T\sigma(t) - f_p\left(X_T\right)\int_0^T D_t\alpha(s)\,d\tilde{W}_s \,\Big|\, \mathcal{F}_t\right]. \tag{65}$$

In particular, if $dX(t) = bX(t)\,dt + \sigma\,dW(t)$ and $r(t) = r > 0$, then $D_t\alpha = D_t(b-r)\sigma^{-1} = 0$ a.e., and

$$\theta(t) = X_0(t)E_Q\left[I_{[p,\infty)}\left(X_T\right)X_T \mid \mathcal{F}_t\right]. \tag{66}$$

For any $t \leq T$, the value of the derivative at time t is determined as the value $V_\theta(t)$ of the portfolio $\theta(t)$.

When X_T follows a diffusion process, (66) leads to the classical Black-Scholes formula applying Markovian arguments. However, in case of boundary-linked assets markets, as these assets follow a boundary value stochastic differential equation, the expectation in (66) cannot be computed using Markovian arguments and numerical resolution methods are required.

In order to compute the portfolio $\{\theta_t\}$, associated to a given a realization of the underlying processes $\{(X_t, W_t)\}$, we propose the use of a Monte Carlo simulation-based estimation of (66) using independently generated realizations of the process $\{X_t\}$ conditioned to the information set $\mathcal{F}_t$, which is computed using the wavelet-collocation approach presented in Section 3. Three steps are involved. For each dyadic point $t_{i,n} = 2^{-n}i \in [0,T]$ with $i \in \mathbb{Z}$, we simulate M independent realizations of the Brownian motion, denoted by $W^j_{t_{i,n}}$ for all $j = 1,\ldots,M$, such that $W^j_t = W_t$, for any dyadic point $t \in [0, t_{i,n}]$. The second step of the algorithm consists of solving, for each $j = 1,\ldots,M$, the following system of boundary value stochastic differential equations:

$$dX^j_t = bX^j_t\,dt + \sigma dW^j_t, \tag{67}$$

$$\beta\left(X^j\right) = \rho, \tag{68}$$

with the additional constraints $X_t^j = X_t$, for any dyadic point $t \in [0, t_{i,n}]$. In particular, we compute the solution of these problems $\{X_t^j\}_{j=1}^M$ by means of the wavelet-collocation algorithm presented in Section 3. In the third and final step, we compute portfolio (66) at each $t_{i,n} \in [0,T]$ as

$$\theta^M(t_{i,n}) = \exp(rt_{i,n}) \frac{1}{M} \sum_{j=1}^{M} I_{[p,\infty)}\left(X_T^j\right) X_T^j. \tag{69}$$

For example, Figure 4 shows the numerical simulations of X_t conditioned to the available information at $t_{i,n} = 0.25$, when X_t follows the boundary-valued stochastic differential equation given in Example I and $M = 50$. Figure 5 shows the path of portfolio (66) computed as (69) with $r = 0.2$ and $p = 0.25$.

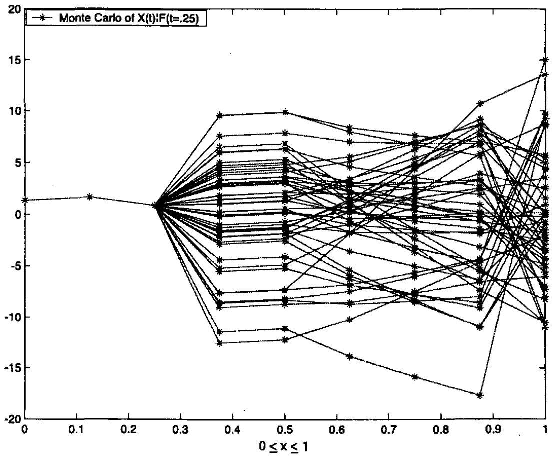


Figure 4. Monte Carlo simulations of $X_t \mid \mathcal{F}_{t=0.25}$.

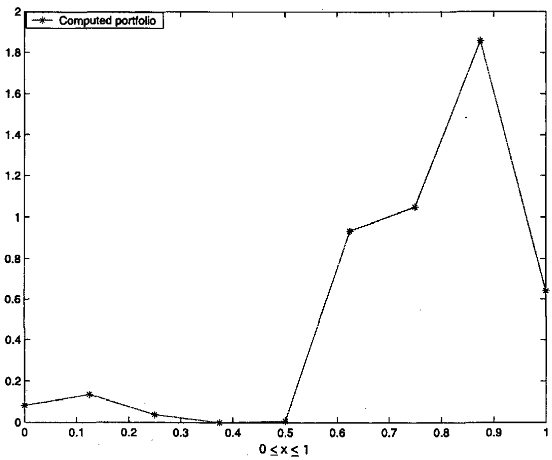


Figure 5. Path of portfolio (66), with $r = 0.2$ and $p = 0.25$.

APPENDIX A
PROOFS

(A) Proof of Theorem 1

We will use the following theorem.

THEOREM 5. *Let B be a Banach space, $\{V_n\} \subset B$ a sequence of increasing linear subspaces, and Π_{V_n} a sequence of continuous projections converging pointwise to the identity operator on B. Let T define a (nonlinear) operator in B. If $(1-T)u = 0$ has a solution u^0, T is continuously Fréchet differentiable at u^0 and $(1 - T'_{u^0})u = 0$ has only the trivial solution in B, then u^0 is unique in some sphere $B(u^0, \delta) = \{u \in B : \|u - u^0\| \le \delta\}$ for some $\delta > 0$, and there exists an integer M such that for all $n > M$ the equation $\Pi_{V_n}\{(1-T)u\} = 0$ has a unique solution $u_n \in V_n \cap B(u^0, \delta)$. Moreover, $\exists K > 0$ such that*

$$\left\|u_n - u^0\right\| \le K \left\|\Pi_{V_n} u^0 - u^0\right\|. \tag{70}$$

PROOF. See, e.g., [36, Theorem 5]). ∎

Using the properties of the Green function and the continuity of b, the functional T is continuous relative to the uniform norm on a neighborhood of $u^0 = G(x^0)$. Since for each realization of the white noise process $(1-T)u = 0$ can be seen as an equation in $C_0([0,T], \mathbb{R}^d)$, we will consider the equation $\Pi_{V_n}(I-T)u_n = 0$ in V_n.

First, we check the continuous Fréchet differentiability of T. For any $u \in B(u^0, \delta)$ define $h = u - u^0$. Notice that $\mathcal{N}$ contains all line segments in $\mathbb{R}^{R+1}$ such as $\{u^0 + \theta h : \theta \in [0,1]\}$, since

$$x(t) - x^0(t) = \int_0^T D_t G(t,s) h(s)\, ds, \tag{71}$$

with $\|x - x^0\|_{L_\infty} < \varepsilon$ whenever δ is small enough, using that

$$\chi := \operatorname*{esssup}_{t\in[0,T]} \int_0^T |D_t G(t,s)|\, ds < \infty, \tag{72}$$

where esssup denotes the essential supremum with respect to the Lebesgue measure (i.e., esssup f is the smallest number $a \in \mathbb{R}$ for which f only exceeds a on a set of measure zero).

Recall that $u^0 = G(x^0)$. We will see that the Fréchet derivative T at $u^0(t) = Dx^0(t)$ respect to the direction $h = (u - u^0)$ is given by

$$T'_{u^0}(h)(t) = \left(D_u b\left(t, u^0\right) + D_u \sigma\left(t, u^0\right) \dot{W}_t\right) \int_0^T D_t G(t,s) h(s)\, ds, \tag{73}$$

and the error term is given by

$$\begin{aligned}\epsilon_{u^0}(u)(t) &= |||h|||^2 \int_0^T (1-\theta) b''\left(t, u^0(t) + \theta h(t)\right)\, dt \\ &\quad + |||h|||^2 \int_0^T (1-\theta)\sigma''\left(t, u^0(t) + \theta h(t)\right)\, dW(t),\end{aligned} \tag{74}$$

b'', σ'' being the second directional derivatives of $b(t,\cdot)$, $\sigma(t,\cdot)$, respectively, in the direction $h/|||h|||$, and $|||h|||^2 = \sum_{r=1}^R \|h_r\|^2$. Clearly $\|\epsilon_{u^0}(u)\|_{L_\infty} \le c_1 \|u - u^0\|^2_{L_\infty}$, where c_1 is the maximum between χ and $\sup\{b''(t,x) + \sigma''(t,x) W_T\}$ over all directions on $\mathrm{cl}(\mathcal{N})$, the closure of $\mathcal{N}$, which is finite with probability one as $\Pr(W_T = \infty) = 0$ for finite T.

Notice also that $T'_{u^0}(h)(t)$ can be expressed in the original domain as the operator

$$T'_{x^0}(x) = \left(D_x b\left(t, x^0(t)\right) + D_x \sigma\left(t, x^0(t)\right) \dot{W}_t\right) Dx. \tag{75}$$

Since $\det\{(I - D_x b(t, x^0(t)) - D_x\sigma(t, x^0(t))\dot{W}_t)\} \neq 0$, almost surely, for all $t \in [0, T]$, there exists a unique trivial solution for

$$\left(I - D_x b\left(t, x^0(t)\right) - D_x \sigma\left(t, x^0(t)\right) \dot{W}_t\right) Dx = 0, \tag{76}$$

with $\alpha(x) = c$. This implies the same result for $(I - T'_{u^0})u = 0$, and hence assumptions of Theorem 5 are satisfied.

Thus, there exists an integer $M > 0$ such that, for all $n > M$, a solution $u_n \in V_n$ exists and is unique in the same sphere. Moreover, there exists a constant $c > 0$ such that $u_n = Dx_n$, $u^0 = Dx^0$, and

$$\left\|u_n - u^0\right\|_{L_\infty} \leq c \left\|\Pi_{V_n} u^0 - u^0\right\|_{L_\infty}. \tag{77}$$

By the Banach-Steinhaus theorem, for all $u \in V_n$,

$$\begin{aligned} \left\|\Pi_{V_n} u^0 - u^0\right\|_{L_\infty} &= \left\|\Pi_{V_n}\left(u^0 - u\right) - \left(u - u^0\right)\right\|_{L_\infty} = \left\|(1 - \Pi_{V_n})\left(u^0 - u\right)\right\|_{L_\infty} \\ &\leq c' \inf\left\{\left\|u^0 - u\right\|_{L_\infty} : u \in V_n\right\} = O\left(2^{-n}\right), \end{aligned} \tag{78}$$

where the rate $O(2^{-n})$ follows from Assumption A.1.

The result follows noticing that $\|Dx_n - Dx^0\|_{L_\infty} = \|u_n - u^0\|_{L_\infty}$, and

$$\left\|x_n - x^0\right\|_{L_\infty} \leq \left\|G^{-1}\right\|_{L_\infty} \left\|u_n - u^0\right\|_{L_\infty} \tag{79}$$

using that $x_n - x^0 = G^{-1}(u_n - u^0)$.

(B) Proof of Theorem 2

The problem of interpolation in V_n at points $t_{n,i} = 2^{-n} i$ can be reduced to solve the problem $g_0(i) = x(t_{n,i})$ in $g_0 \in V_0$ and then take $g_n(t) = 2^{n/2} g_0(2^n t)$. Therefore, assume that $g_0(t) = \sum_{k\in\mathbb{Z}} \theta_k \phi(t - k)$ solves this problem, i.e.,

$$\sum_{k\in\mathbb{Z}} \theta_k \phi(i - k) = x\left(t_{n,i}\right). \tag{80}$$

Clearly, a unique solution exists since $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ are linearly independent functions. To simplify the notation, we denote $x_i = x(t_{n,i})$, and hence $\sum_{k\in\mathbb{Z}} \theta_k \phi(i - k) = x_i$. This is a convolution equation that we will solve in the spectral domain. Let us define the discrete Fourier transform of ϕ by

$$\tilde{\Phi}(\omega) = \sum_{k\in\mathbb{Z}} \phi(k) e^{-ik\omega}. \tag{81}$$

The Poisson formula states that $\tilde{\Phi}(\omega) = \sum_{k\in\mathbb{Z}} \Phi(\omega + 2\pi k)$. If ϕ is regular of at least order 1, this series converges uniformly on compact sets. Furthermore, as $\tilde{\Phi}(\omega) > 0$ a.e. for $\omega \in [0, 2\pi]$, the inverse has a Fourier expansion $(1/\tilde{\Phi}(\omega)) = \sum_{k\in\mathbb{Z}} \beta_k e^{-ik\omega}$ where $b := \sum_{k\in\mathbb{Z}} |\beta_k| < \infty$, by the Wiener-Lévy theorem. Thus, we can explicitly evaluate the coefficients $\{\theta_k\}$ as

$$\theta_k = \sum_{i\in\mathbb{Z}} \beta_{k-i} x_i. \tag{82}$$

Obviously,

$$\sum_{k\in\mathbb{Z}} |\theta_k|^2 = \sum_{k\in\mathbb{Z}} \left|\sum_{i\in\mathbb{Z}} \beta_{k-i} x_i\right|^2 \leq b^2 \sum_{i\in\mathbb{Z}} |x_i|^2 = b^2 \sum_{i\in\mathbb{Z}} \left|x\left(t_{n,i}\right)\right|^2, \tag{83}$$

with $\sup_{n>1} \sum_{i\in\mathbb{Z}} |x(t_{n,i})|^2 < \infty$ as x is continuous with compact support.

Next, we will prove that

$$\|\Gamma_{V_n}(x)\|_{L_2} \leq b\|x\|_n, \tag{84}$$

where $\|x\|_n = (2^{-n}\sum_{i\in\mathbb{Z}}|x(t_{n,i})|^2)^{1/2}$, applying Schwartz's arguments as in [37].

Notice that $\|g_n\|_{L_2} = 2^{-n}\|g_0\|_{L_2} = 2^{-n}\|\mathcal{F}(g_0)\|_{L_2}$, where $\mathcal{F}(g_0)(\omega)$ is the continuous Fourier transformed of g_0. We will prove that $\|\mathcal{F}(g_0)\|_{L_2}^2 = \sum_{k\in\mathbb{Z}}|\theta_k|^2$ and the result follows. Let us define $\tilde{c}(\omega) = \sum_{k\in\mathbb{Z}}\theta_k e^{-ik\omega}$, then

$$\begin{aligned}
\|\mathcal{F}(g_0)\|_{L_2}^2 &= \int_{\mathbb{R}}\left|\mathcal{F}\left(\sum_{k\in\mathbb{Z}}\theta_k\phi_{0,k}\right)(\omega)\right|^2 d\omega = \int_{\mathbb{R}}\left|\Phi(\omega)\left(\sum_{k\in\mathbb{Z}}\theta_k e^{-ik\omega}\right)\right|^2 d\omega \\
&= \int_{\mathbb{R}}|\Phi(\omega)\tilde{c}(\omega)|^2\, d\omega = \sum_{k\in\mathbb{Z}}\int_{2k\pi}^{2(k+1)\pi}|\Phi(\omega)\tilde{c}(\omega)|^2\, d\omega \\
&= \int_0^{2\pi}|\tilde{c}(\omega)|^2\left|\sum_{k\in\mathbb{Z}}\Phi(\omega+2\pi k)\right|^2 d\omega \\
&= \int_0^{2\pi}|\tilde{c}(\omega)|^2\, d\omega = \sum_{k\in\mathbb{Z}}|\theta_k|^2,
\end{aligned} \tag{85}$$

as $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ is orthonormal if and only if $\sum_{k\in\mathbb{Z}}|\Phi(\omega+2\pi k)|^2 = 1$ a.e., for details see [23]. Hence, we have that $\|\Gamma_{V_n}(x)\|_{L_2}^2 \leq b^2\|x\|_n^2$.

Defining $x_n = \Pi_{V_n}(x)$, we have that $\Gamma_{V_n}(x_n) = x_n$ since $x_n \in V_n$ and has compact support. And as a consequence,

$$\begin{aligned}
\|\Gamma_{V_n}(x) - x\|_{L_2} &= \|\Gamma_{V_n}(x_n - x) + x_n - x\|_{L_2} \leq b^2\|x_n - x\|_n + \|x_n - x\|_{L_2} \\
&= b^2\|\Pi_{V_n}(x) - x\|_n + \|\Pi_{V_n}(x) - x\|_{L_2}.
\end{aligned} \tag{86}$$

Moreover, as for all $x \in W_2^r(\mathbb{R})$, with $r \geq 1$,

$$\|x\|_n^2 \leq C\left\{\int_{-2^n\pi}^{2^n\pi}|\mathcal{F}(x)(\omega)|^2\, d\omega + 2^{-nr}\|x\|_{W_2^r}^2\right\}, \tag{87}$$

see [38], the result follows applying the same bound to $\|\Pi_{V_n}(x) - x\|_n^2$.

(C) Proof of Proposition 4

Let X_n be the solution to (43), and $\tilde{X}_n$ be the solution to (44), i.e., the proposed algorithm. We will prove that $E[\|X_n - \tilde{X}_n\|_\infty^2] = O(h_n^2)$ and the result follows.

Consider first the autonomous case. By assumption $b, \sigma \in C^2(\mathcal{N})$. Define the operators,

$$L^0 = b\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}, \qquad L^1 = \sigma\frac{\partial}{\partial x}. \tag{88}$$

By the Wagner and Platen expansion (see [29], for a review), X_n satisfies the equations

$$A_{i,n}(X_n) = R_{i,n}(X_n), \tag{89}$$

for all $i \in \mathbb{Z}$ such that $t_{i,n} = 2^{-n}i \in [0,T]$, where

$$A_{i,n}(X_n) = X_n(t_{i,n}) - X_n(t_{i-1,n}) - h_n b(X_n(t_{i-1,n})) - \sigma(X_n(t_{i-1,n}))\left(W_{t_{i,n}} - W_{t_{i-1,n}}\right)$$

$$- L^1\sigma(X_n(t_{i-1,n})) \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^{s} dW_z\, dW_s, \tag{90}$$

$$R_{i,n}(X_n) = \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^{s} L^0 b(X_n(z))\, dz\, ds + \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^{s} L^1 b(X_n(z))\, dW_z\, ds$$

$$+ \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^{s} L^0\sigma(X_n(z))\, dz\, dW_s$$

$$+ \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^{s} \int_{t_{i-1,n}}^{z} L^0 L^1\sigma(X_n(u))\, du\, dW_z\, dW_s \tag{91}$$

$$+ \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^{s} \int_{t_{i-1,n}}^{z} L^1 L^1\sigma(X_n(u))\, dW_u\, dW_z\, dW_s.$$

Let $A_n(X_n) = R_n(X_n)$ denote this system of nonlinear equations, where $E[\|R_n(X_n)\|_\infty^2] = O(h_n^2)$.

On the other hand, $\tilde{X}_n$ satisfies system (44); i.e., $A_n(\tilde{X}_n) = 0$. Then,

$$R_n(X_n) = A_n(X_n) = A_n\left(\tilde{X}_n\right) + DA_{\varphi_n}\left(X_n - \tilde{X}_n\right) = DA_{\xi_n}\left(X_n - \tilde{X}_n\right), \tag{92}$$

where DA_{φ_n} is the Fréchet derivative at some intermediate point φ_n. Since $\|DA_{\varphi_n}(\cdot)\|_\infty^{-1} \geq \varepsilon > 0$ uniformly, it is satisfied that

$$E\left[\left\|X_n - \tilde{X}_n\right\|_\infty^2\right] = O\left(E\left[\|R_n(X_n)\|_\infty^2\right]\right), \tag{93}$$

and the result follows. For nonautonomous systems the argument is analogous.

APPENDIX B
MATLAB CODE FOR EXAMPLE I

The proposed algorithm can be efficiently implemented using MATLAB 6.5 (`http://www.mathworks.com`). The wavelet toolbox in MATLAB provides a comprehensive collection of routines for applying wavelet approximations. We present the code for solving Example I, the simplest example, because it provides enough information to understand how construct a code for other problems.

The first two steps of the implementation are: the computation of the Wiener process and the Daubechies wavelets. A Wiener process is the sum of zero mean Gaussian variables, computed with the help of `randn(1,dimension)`, an internal MATLAB function that produces a 1-by-dimension matrix with random entries, chosen from a normal distribution with mean zero, variance one, and standard deviation one (we also call `randn('state',100)` to reset the generator to the state 100). To generate Daubechies wavelet, we use `wavefun` from MATLAB's wavelet toolbox. These wavelets have no explicit expression. Mallat's cascade algorithm gives a constructive and efficient recipe for defining compactly supported wavelets such as Daubechies (for details see [23]). By typing `waveinfo('db')`, at the MATLAB command prompt, you can obtain a survey of the main properties of this family.

The next step consists of solving the system of equations (26) and (27) in $\theta_{n,k} \in \mathbb{R}$ where $b(t, X_t) = 0$, $\sigma(t, X_t) = 1$, $\alpha(X_t) = X_{1/2} + X_1$, and $c = 0$. The specific algorithm used for solving this system depends upon the structure of the coefficient matrix A. For this example, we use the backslash operator to solve simultaneous linear equations. In general, the optimization toolbox can solve systems of nonlinear equations (see the reference page in the online MATLAB documentation for more information).

```
% EXAMPLE I: dx(t)=dW(t), x(.5)+x(1)=0, t in [T1,T2]
% Solution x(t)=-.5*(W(.5)+W(1))+W(t)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Parameters
mu=0;
sigma=1;
T1=0;
T2=10;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%First step: Definition wavelet to approximate
N=3; % N, Daubechies' order
n=4; % n, resolution level
wname='db3'; % wavelet name
[phi,psi,x] = WAVEFUN(wname,n);
step = x(2)-x(1); %size of approximation step
dimension=length(x);
h=2^(-n);
n_vars=T2*2^n-1-(T1*2^n+2-2*N)+1; % number of
coefficients thetas
% Definition of Wiener process
randn('state',100)
dW= sqrt(h)*randn(1,dimension); % Wiener increments
W = cumsum(dW); % discretized Wiener path
W=[0;W'];
% Analytical solution
exact_x= -.5*(W(.5*2^n+1)+W(1*2^n+1))+W;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Second step: Computation of K=F(thetas), system of equations (26) and (27)
% First: Equation (26)
for i=T1*2^n+2-2*N:T2*2^n-1,
inner_prod1(:,i-(T1*2^n+2-2*N)+1)=zeros(dimension-1,1);
 for l=1:dimension-1,
 phi_i1(l)=0;
 if (((2^n)*x(l+1)-i>=0) & (((2^n))*x(l+1)-i<=2*N-1) ),
 phi_i1(l) = (2^(n/2)) * phi(((2^n)*x(l+1)-i)*2^n+1);
 end
 end
 inner_prod1(:,i-(T1*2^n+2-2*N)+1) = phi_i1(:);
inner_prod2(:,i-(T1*2^n+2-2*N)+1)=zeros(dimension-1,1);
 for l=1:dimension,
 phi_i2(l)=0;
 if (((2^n)*x(l)-i>=0) & (((2^n))*x(l)-i<=2*N-1) ),
 phi_i2(l) = (2^(n/2)) * phi(((2^n)*x(l)-i)*2^n+1);
 end
 end
 inner_prod2(:,i-(T1*2^n+2-2*N)+1) = -phi_i2(1:dimension-1)';
end
R=inner_prod1+inner_prod2;
% Second: Computation of last equation of system (27), in this case x(.5)+x(1)=C
aux_K1=zeros(n_vars,1);
aux_K2=zeros(n_vars,1);
```

```
for j=T1*2^n+2-2*N:T2*2^n-1,
 if (((2^n)*.5-j>=0) & ((2^n)*.5-j<=2*N-1)),
aux_K1(j-(T1*2^n+2-2*N)+1)=(2^(n/2))*phi(((2^n)*.5-j)*2^n+1); end
end
for j=T1*2^n+2-2*N:T2*2^n-1,
 if (((2^n)-j>=0) & ((2^n)-j<=2*N-1)),
aux_K2(j-(T1*2^n+2-2*N)+1)=(2^(n/2))*phi(((2^n)-j)*2^n+1); end
end
K=[R;aux_K1'+aux_K2'];
% The right side of system (27)
b=[dW(1:dimension-1)';0];
% Third: Solve Kx=b. Note that in this example it is linear. The backslash
operator employs different algorithms to handle different kinds of coefficient
matrices.
theta=K\b;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Informative results
error_sist=K*theta-b;
% Approximation error
norm(error_sist,inf)
% Condition number of K
cn=cond(K)
% Digits of precision that we can expect to lose during the process
d=log(cn)/log(10)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Third step: To compute the numerical solution given the optimal
% coefficients thetas.
x=0:2^(-n):2;
aprox=zeros(length(x),1);
for k = T1*2^n+2-2*N:T2*2^n-1,
 for i=1:length(x),
 phijk(i,k-(T1*2^n+2-2*N)+1)=0;
 if (((2^n)*x(i)-k>=0) & (((2^n))*x(i)-k<=2*N-1)),
phijk(i,k-(T1*2^n+2-2*N)+1)=(2^(n/2))*phi(((2^n)*x(i)-k)*2^n+1);
 end
 aprox(x(i)*2^n+1)=aprox(x(i)*2^n+1)+...
theta(k-(T1*2^n+2-2*N)+1).*phijk(i,k-(T1*2^n+2-2*N)+1);
 end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plot the results versus the analytical solution
figure(1)
plot(x,exact_x(1:length(x)),'-*',x,aprox,'v-')
h = legend('Solution','Euler Aprox.',2);
xlabel('0 \leq x \leq 2')
title(' ')
% Display the approximation error
disp(norm(exact_x(1:length(x))-aprox))
```

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